FINITE QUANTUM GROUPS AND QUANTUM PERMUTATION GROUPS

TEODOR BANICA, JULIEN BICHON, AND SONIA NATALE

Abstract. We give examples of finite quantum permutation groups which arise from the twisting construction or as bicrossed products associated to exact factorizations in finite groups. We also give examples of finite quantum groups which are not quantum permutation groups: one such example occurs as a split abelian extension associated to the exact factorization $\mathbb{S}_4 = \mathbb{Z}_4\mathbb{S}_3$ and has dimension 24. We show that, in fact, this is the smallest possible dimension that a non quantum permutation group can have.

1. Introduction

Let $n \geq 1$ be an integer. Recall from Wang's paper [29] that the usual symmetric group $\mathbb{S}_n$ has a free analogue, denoted $\mathbb{S}_n^+$; this is a compact quantum group acting universally on the set $\{1, \dots, n\}$.

We shall work over an algebraically closed base field k of characteristic zero. Let $\mathcal{A}_s(n) = \mathcal{A}_s(n, k)$ be the Hopf algebra corresponding to Wang's quantum permutation group [7]. This is the algebra given by generators u_{ij}, $1 \leq i, j \leq n$, with relations making $(u_{ij})_{i,j}$ a *magic* matrix, that is,

$$u_{ij}u_{ik} = \delta_{jk}u_{ij}, \quad u_{ij}u_{kj} = \delta_{ik}u_{ij}, \quad \sum_{l=1}^{n} u_{il} = 1 = \sum_{l=1}^{n} u_{li}, \tag{1.1}$$

for all $1 \leq i, j, k \leq n$. The algebra $\mathcal{A}_s(n)$ is a cosemisimple Hopf algebra with comultiplication, counit and antipode determined by

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}, \quad \epsilon(u_{ij}) = \delta_{ij}, \quad \mathcal{S}(u_{ij}) = u_{ji}, \quad 1 \leq i, j \leq n. \tag{1.2}$$

By a (finite) *quantum permutation algebra* we shall understand a (finite dimensional) quotient Hopf algebra H of $\mathcal{A}_s(n)$. In particular, a quantum permutation algebra satisfies $\mathcal{S}^2 = \mathrm{id}$. Hence, a finite quantum permutation algebra is cosemisimple and semisimple. Formally, a quantum permutation algebra corresponds to a "quantum permutation group", that is, a quantum subgroup of $\mathbb{S}_n^+$.

The Hopf algebra $\mathcal{A}_s(n)$ is the universal cosemisimple Hopf algebra coacting on the commutative algebra k^n [7]. Thus, a cosemisimple Hopf algebra H is a quantum permutation algebra if and only if there exists a separable commutative faithful (left or right) H-comodule algebra L.

The Hopf algebra $k^{\mathbb{S}_n}$ of functions on the classical symmetric group is a quantum permutation algebra (the maximal commutative quotient of $\mathcal{A}_s(n)$). We have $k^{\mathbb{S}_n} =$

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$\mathcal{A}_s(n)$, for $n = 1, 2, 3$, but not for $n \geq 4$, where the algebra $\mathcal{A}_s(n)$ is not commutative and infinite dimensional. See [29, 7].

Several interesting examples of quantum permutation groups, finite or not, were obtained as twisting deformations of classical groups. Here is a list of such quantum groups:

(1) The twists of $\mathbb{S}_n$ from [6].
(2) The quantum group O_n^{-1}, see [2].
(3) The twists of several subgroups of SO_3, see [1].

Also the nontrivial Hopf algebras studied by Masuoka [22] appear as quantum permutations algebras in [1], including the historical 8-dimensional Kac-Paljutkin example (which is not a twist of a function algebra).

As is well-known, the Cayley representation makes every finite group into a permutation group. This leads naturally to consider the question whether any "finite quantum group" is a "quantum permutation group". In other words:

Question 1.1. *Let H be a finite dimensional cosemisimple Hopf algebra. Is it true that H is a quotient of $\mathcal{A}_s(N)$, for some $N \in \mathbb{N}$?*

Of course, the answer is 'yes' if H is commutative, taking $N = \dim H$. Although less trivial to see, the answer is also 'yes' in the cocommutative case. Moreover, as shown in [7], the cocommutative cosemisimple (finite or not) Hopf algebra quotients of $\mathcal{A}_s(N)$ are exactly the group algebras kG, where G is a quotient of a free product $G_1 * \cdots * G_m$ of transitive abelian groups $G_i \subseteq \mathbb{S}_{n_i}$, with $N = \sum_i n_i$.

Now back to our question, let G be a finite group of order n. Then there is a canonical surjective homomorphism $\mathbb{Z}_n^{*n} \to G$, given by $k^{(g)} \mapsto g^k$. Thus kG is a quotient of $\mathcal{A}_s(N)$, with $N = n^2$.

This also suggests that one could include the condition $N = (\dim H)^2$ in Question 1.1.

In this paper we show that the answer to Question 1.1 is negative in general. More precisely, we give examples of finite dimensional cosemisimple Hopf algebras which are not quantum permutation algebras. Such examples arise as split abelian extensions from exact factorizations of the symmetric groups $\mathbb{S}_4$ and $\mathbb{S}_5$. See Theorem 7.4.

These examples show that the class of finite quantum permutation algebras is not stable under extensions. It turns out that their duals are quantum permutation algebras, so we get that the class of finite quantum permutation algebras is also not stable under duality. An argument involving Drinfeld doubles implies, in addition, that this class is not stable under twisting deformations neither.

We also discuss sufficient conditions on abelian extensions or a twisting deformation of a linear algebraic group in order that they be quantum permutation algebras. This is done in Sections 5 and 8, respectively. Some known examples of quantum permutation algebras turn out to fit into these pictures.

We show that central abelian extensions and certain classes of split extensions, that include cocentral split extensions and split extensions by an abelian group, are quantum permutation algebras. See Theorems 5.1 and 5.2.

As a consequence, we get that if G is a finite group, then the Drinfeld double $D(G)$ and its dual $D(G)^*$ are quantum permutation algebras. We also obtain that a cosemisimple Hopf algebra whose dimension divides p^3 or pqr, where p, q and r are pairwise distinct prime numbers, is a quantum permutation algebra (Proposition

5.8). Other known examples also fit into this picture, like, for instance, some nontrivial Hopf algebras studied by Masuoka [22].

We then look at twisting deformations of a quantum permutation algebra H. We give in Proposition 8.1 a general sufficient condition on a cocycle $\sigma : H \otimes H \to k$ such that the twisted Hopf algebra H^σ is a quantum permutation algebra. The deformations of the symmetric groups in [6] fall into this class.

Let Γ be a finite abelian group and σ a 2-cocycle on Γ. We give further examples of quantum permutation algebras as twisting deformations of linear algebraic groups G, with $\widehat{\Gamma} \subset G \subset \mathrm{Aut}(k_\sigma \Gamma)$. This construction relies on the results of [3]. See Theorem 8.3. The twisted examples from [2] fit into this framework.

In Section 9 we introduce the *quantum permutation envelope*, denoted H_{qp}, of a cosemisimple Hopf algebra as the subalgebra generated by the matrix coefficients of all separable commutative (right and left) coideal subalgebras of H. This is a Hopf subalgebra containing all quantum permutation algebras $A \subseteq H$. When H is finite dimensional, H_{qp} is the maximal quantum permutation algebra contained in H. This provides a method to construct quantum permutation algebras from any finite dimensional cosemisimple Hopf algebra. We determine the quantum permutation envelope for some families of examples.

Conventions. We refer the reader to [24] for the notation and terminology on Hopf algebras used throughout. By a *twisting deformation* of a Hopf algebra H we understand a twist H^σ in the sense of Doi [9], where $\sigma : H \otimes H \to k$ is a convolution invertible normalized 2-cocycle. That is, $H^\sigma = H$ as a coalgebra, with multiplication

$$[x][y] = \sigma(x_1, y_1)\sigma^{-1}(x_3, y_3)[x_2y_2], \quad x, y \in H,$$

where $[x]$ denotes the element $x \in H$, viewed as an element of H^σ.

2. Quantum permutation algebras

Let H be a quantum permutation algebra. As noticed before, we have $\mathcal{S}^2 = \mathrm{id}$ in H. For H finite-dimensional this condition is equivalent to H being separable and/or cosemisimple [15].

Definition 2.1. The *degree* of H, denoted $\mathrm{d}(H)$, is the smallest $n \geq 1$, such that H is a quotient Hopf algebra of $\mathcal{A}_s(n)$.

As pointed out in the Introduction, if $H = k^G$, where G is a finite group, then $\mathrm{d}(H) \leq |G|$, while if $H = kG$, then $\mathrm{d}(H) \leq |G|^2$ (see [7, Proposition 5.2]).

Let $f(u_{ij}) =: x_{ij} \in H$, where $f : \mathcal{A}_s(n) \to H$ is a surjective Hopf algebra map.

For each $j = 1, \dots, n$, consider the subspace L^j of H spanned by x_{ij}, $1 \leq i \leq n$. By (1.1) and (1.2), each L^j is a commutative separable left coideal subalgebra of H. Similarly, the subspace $R^j = \mathcal{S}(L^j)$ of H spanned by x_{ji}, $1 \leq i \leq n$, is a commutative separable right coideal subalgebra.

In particular, H is generated as an algebra by its commutative separable left coideal subalgebras $L^1, \dots, L^n$.

Remark 2.2. If H_1, H_2 are quantum permutation algebras, then so is their free product $H_1 * H_2$ (with block-diagonal magic matrix) [29, 4]. It follows that any cosemisimple Hopf algebra H such that H is generated as an algebra by a finite number of quantum permutation algebras is itself a quantum permutation algebra.

Further, if $H = k[H_1, \dots, H_r]$ is generated by the quantum permutation algebras $H_1, \dots, H_r$, then $\mathrm{d}(H) \leq \mathrm{d}(H_1) + \cdots + \mathrm{d}(H_r)$.

Theorem 2.3. *Let H be a finite dimensional cosemisimple Hopf algebra. Then H is a quantum permutation algebra if and only if H is generated, as an algebra, by the matrix coefficients of its commutative left (or right) coideal subalgebras.*

Proof. We have already proved the 'only if' implication. To prove the converse, suppose $L \subseteq H$ is a left (or right) coideal subalgebra. It is known that H is free as a (left or right) L-module under multiplication (see [16] or, more generally, [28]). This implies that L is a separable algebra, by [17].

Let $H[L]$ be the subalgebra generated by the matrix coefficients of L. Then $H[L]$ is a subbialgebra of H, and therefore a Hopf subalgebra, because it is finite dimensional. Suppose L is commutative. Since, by definition, $H[L]$ coacts faithfully on L, then it is a quantum permutation algebra [7].

Since H is finite dimensional, the assumption implies that H is generated by a finite number of the Hopf subalgebras $H[L]$. Hence H is a quantum permutation algebra, by Remark 2.2. This finishes the proof of the theorem. □

If H is a quantum permutation algebra, then so are H^{op} and H^{cop}. For a finite dimensional Hopf algebra H, the Drinfeld double $D(H)$ is generated as an algebra by $H^{*\,\mathrm{cop}}$ and H. Moreover, $D(H)$ is cosemisimple if H (and therefore also H^*) is cosemisimple. Then we get:

Corollary 2.4. *Let H be a finite dimensional cosemisimple Hopf algebra. If H and H^* are quantum permutation algebras, then the Drinfeld double $D(H)$ is a quantum permutation algebra and we have $\mathrm{d}(D(H)) \leq \mathrm{d}(H) + \mathrm{d}(H^*)$.*

In particular, every Drinfeld double $D(G)$ of a finite group algebra kG is a quantum permutation algebra of degree at most $|G|(1 + |G|)$.

We shall see later (Corollary 5.6) that $D(G)^*$ is also a quantum permutation algebra.

3. Hopf algebra extensions

Recall that an *exact sequence* of finite dimensional Hopf algebras is a sequence of Hopf algebra maps

$$k \to A \xrightarrow{\iota} H \xrightarrow{\pi} \overline{H} \to k, \tag{3.1}$$

where H is finite dimensional, such that ι is injective, π is surjective and, identifying A with a Hopf subalgebra of H, we have

$$A = H^{\mathrm{co}\,\pi} = \{h \in H |\, (\mathrm{id} \otimes \pi)\Delta(h) = h \otimes 1\}. \tag{3.2}$$

See [23, Definition 1.4] for details. (In particular, condition (3.2) is equivalent to $H/HA^+ = \overline{H}$, where $A^+ = \ker \epsilon_A$, and we identify $\overline{H}$ as a quotient Hopf algebra of H.) We shall say in this case that H is an *extension* of A by $\overline{H}$.

Remark 3.1. Suppose H is finite dimensional and $A \subseteq H^{\mathrm{co}\,\pi}$ is a Hopf subalgebra. Let us point out that in this case exactness of the sequence (3.1) is equivalent to the condition $\dim A \dim \overline{H} = \dim H$.

The following result on Hopf algebra extensions will be used repeatedly.

Theorem 3.2. *Suppose H is finite dimensional cosemisimple. Assume in addition that:*

(i) *A is a quantum permutation algebra, and*

(ii) *$\overline{H}$ is generated as an algebra by a finite subset X such that, for all $x \in X$, there exists a commutative left coideal subalgebra L^x of H with $x \in \pi(L^x)$.*

Then H is a quantum permutation algebra. Moreover, we have $\mathrm{d}(H) \leq \mathrm{d}(A) + \sum_{1 \neq x \in X} \dim L^x$.

Note that the same statement holds true replacing *left* by *right*.

Proof. For each $x \in X$, let $H^x \subseteq H$ be the subalgebra generated by the subcoalgebra C^x of matrix coefficients of L^x. Thus H^x is a Hopf subalgebra, and we have $L^x \subseteq H^x$. Since L^x is commutative, then H^x is a quantum permutation algebra, and $\mathrm{d}(H^x) \leq \dim L^x$, by construction.

Let $\tilde{H}$ be the subalgebra generated by A and H^x, $1 \neq x \in X$. This is a Hopf subalgebra of H that contains A. Moreover, since every element $1 \neq x \in X$ belongs to $\pi(L^x) \subseteq \pi(\tilde{H})$, and X generates $\overline{H}$, then $\pi|_{\tilde{H}} : \tilde{H} \to \overline{H}$ is surjective.

On the other hand, by exactness of the sequence $k \to A \to H \xrightarrow{\pi} \overline{H} \to k$, we have $H^{\operatorname{co}\pi} = A$. Thus $\tilde{H}^{\operatorname{co}\pi|_{\tilde{H}}} = H^{\operatorname{co}\pi} \cap \tilde{H} = A$, since $A \subseteq \tilde{H}$. Therefore the sequence $k \to A \to \tilde{H} \to \overline{H} \to k$ is also exact. Then $\tilde{H} = H$, since they have the same finite dimension (see Remark 3.1). Thus H is generated as an algebra by the quantum permutation algebras A, H^x, $1 \neq x \in X$. Hence H is a quantum permutation algebra, with the claimed bound for $\mathrm{d}(H)$, by Remark 2.2. □

4. Matched pairs of groups

Let (F, Γ) be a *matched pair* of finite groups. That is, F and Γ are endowed with actions by permutations $\Gamma \xleftarrow{\triangleleft} \Gamma \times F \xrightarrow{\triangleright} F$ such that

$$s \triangleright xy = (s \triangleright x)((s \triangleleft x) \triangleright y), \quad st \triangleleft x = (s \triangleleft (t \triangleright x))(t \triangleleft x), \tag{4.1}$$

for all $s, t \in \Gamma$, $x, y \in F$.

Given finite groups F and Γ, providing them with a pair of compatible actions is equivalent to giving a group G together with an *exact factorization* $G = F\Gamma$: the relevant actions are determined by the relations $gx = (g \triangleright x)(g \triangleleft x)$, $x \in F$, $g \in \Gamma$.

Consider the left action of F on k^Γ, $(x.f)(g) = f(g \triangleleft x)$, $f \in k^\Gamma$, and let $\sigma : F \times F \to (k^*)^\Gamma$ be a normalized 2-cocycle. Dually, consider the right action of Γ on k^F, $(w.g)(x) = w(x \triangleright g)$, $w \in k^F$, and let $\tau : \Gamma \times \Gamma \to (k^*)^F$ be a normalized 2-cocycle.

Under appropriate compatibility conditions between σ and τ, the vector space $H = k^\Gamma \otimes kF$ becomes a (semisimple) Hopf algebra, denoted $H = k^\Gamma {}^\tau\#_\sigma kF$, with the crossed product algebra structure and the crossed coproduct coalgebra structure (see [23, Section 1]). For all $g, h \in \Gamma$, $x, y \in F$, we have

$$(e_g \# x)(e_h \# y) = \delta_{g \triangleleft x, h}\, \sigma_g(x, y) e_g \# xy, \tag{4.2}$$

$$\Delta(e_g \# x) = \sum_{st=g} \tau_x(s, t)\, e_s \#(t \triangleright x) \otimes e_t \# x, \tag{4.3}$$

where $\sigma_s(x, y) = \sigma(x, y)(s)$ and $\tau_x(s, t) = \tau(s, t)(x)$, $s, t \in \Gamma$, $x, y \in F$.

Let $\pi = \epsilon \otimes \mathrm{id} : H = k^\Gamma {}^\tau\#_\sigma kF \to kF$ denote the canonical projection. We have an exact sequence of Hopf algebras $k \to k^\Gamma \to H \xrightarrow{\pi} kF \to k$. Moreover, every Hopf algebra H fitting into an exact sequence of this form is isomorphic to $k^\Gamma {}^\tau\#_\sigma kF$

for appropriate compatible actions and cocycles σ and τ. Equivalence classes of such extensions associated to a fixed matched pair (F, Γ) form an abelian group $\mathrm{Opext}(k^\Gamma, kF)$, whose unit element is the class of the *split* extension $k^\Gamma \# kF$.

Remark 4.1. Suppose $k = \mathbb{C}$ is the field of complex numbers. Then H is a Hopf C^*-algebra (often called *Kac algebra*), that is, it is a C^*-algebra such that all structure maps are C^*-algebra maps. [14, 21].

Remark 4.2. Let $H = k^\Gamma {}^\tau\#_\sigma kF$ be a bicrossed product. Let $F' \subseteq F$ be a subgroup, and consider the subgroup $\Gamma' \subseteq \Gamma$ consisting of all elements $g \in \Gamma$ such that $g \triangleright F' = F'$. Then (Γ', F') is a matched pair by restriction. Indeed, if $s \in \Gamma'$, $x, y \in F'$, then it follows from the compatibility between $\triangleright$ and $\triangleleft$ that

$$(s \triangleleft x) \triangleright y = (s \triangleright x)^{-1}(s \triangleright xy) \in F',$$

whence $s \triangleleft x \in \Gamma'$. Hence Γ' is F'-stable, which implies the claim.

In particular, if $F' \subseteq F$ is a subgroup stable under the action $\triangleright$, then (Γ, F') is a matched pair by restriction, and it follows from formulas (4.2) and (4.3) that the bicrossed product $k^\Gamma {}^{\tau'}\#_{\sigma'} kF'$ is naturally a Hopf subalgebra of H, where σ' is the restriction of σ to F', and $\tau'_x(g,h) = \tau_x(g,h)$, for all $x \in F'$, $g, h \in \Gamma$.

Observe that if $F' \subseteq F$ is the largest subgroup acting trivially on Γ, then F' is Γ-stable, by (4.1). The Hopf subalgebra $k^\Gamma {}^{\tau'}\#_{\sigma'} kF'$ is in this case a central extension.

5. Quantum permutation algebras obtained from matched pairs of groups

Consider a bicrossed product $H = k^\Gamma {}^\tau\#_\sigma kF$. We shall give in this section sufficient conditions in order for H to be a quantum permutation algebra. These include the following cases:

(1) H is a central abelian extension (that is, k^Γ central in H).
(2) H is a split abelian extension (that is, $\sigma = 1$, $\tau = 1$) and F is generated by its abelian Γ-stable subgroups. (In particular, this is true when F is abelian or the action $\triangleright\colon \Gamma \times F \to F$ is trivial.)

The result for Case (1) is a consequence of Theorem 3.2:

Theorem 5.1. *Let $H = k^\Gamma {}^\tau\#_\sigma kF$ and suppose that k^Γ is central in H. Then H is a quantum permutation algebra and we have* $\mathrm{d}(H) \leq |\Gamma|\,|F|^2$.

Proof. The assumption that k^Γ is central implies that the action $\triangleleft\colon \Gamma \times F \to \Gamma$ is trivial.

Let $x \in F$ and let $\langle x \rangle \subseteq F$ denote the cyclic subgroup generated by x. Consider the subspace $L^x = k^\Gamma \# \langle x \rangle \subseteq H$. It follows from (4.2) and (4.3) that L^x is a left coideal subalgebra of H. As an algebra, $L^x = k^\Gamma \#_{\overline{\sigma}} k\langle x \rangle$ is a crossed product with respect to the trivial action $k\langle x \rangle \otimes k^\Gamma \to k^\Gamma$ and the 2-cocycle $\overline{\sigma} = \sigma|_{\langle x \rangle \times \langle x \rangle}$. Therefore L^x is a commutative left coideal subalgebra of dimension $|x||\Gamma|$.

It is clear that $x \in \pi(L^x)$, for all $x \in F$. Hence H is a quantum permutation algebra, by Theorem 3.2. Moreover, we have that $\mathrm{d}(H) \leq \sum_{x \in F} \dim L_x = |\Gamma| \sum_{x \in F} |x| \leq |\Gamma|\,|F|^2$, as claimed. □

Consider next a *split* abelian extension $H = k^\Gamma \# kF$. It follows from (4.2) and (4.3) that for any subgroup $T \subseteq F$ such that T is stable under the action $\triangleright$ of Γ, the group algebra $kT \simeq 1 \# kT$ is a right coideal subalgebra of H.

Theorem 5.2. *Let $H = k^\Gamma \# kF$ be a split abelian extension. Suppose F is generated by its abelian Γ-stable subgroups. Then H is a quantum permutation algebra. Furthermore, we have* $\mathrm{d}(H) \leq |\Gamma||F|^2$.

Proof. We may assume that $|F|, |\Gamma| > 1$. Let $T_1, \dots, T_s$, be abelian Γ-stable subgroups of F, with $F = \langle T_1, \dots, T_s \rangle$. Then $kT_i \simeq 1 \# kT_i$ are commutative right coideal subalgebras of H, and $x \in \pi(kT_i)$, for all $x \in T_i$. It follows from Theorem 3.2 that H is a quantum permutation algebra. Moreover, we have

$$\mathrm{d}(H) \leq |\Gamma| + \sum_{x \in \cup_i T_i} |T_i^x| \leq |\Gamma| + |F| \, | \cup_i T_i| \leq |\Gamma| + |F|^2 \leq |\Gamma||F|^2,$$

where in the first inequality, T_i^x denotes a choice of one the subgroups T_i such that $x \in T_i$. □

Remark 5.3. Let $T \subseteq F$ be any abelian subgroup. Consider the subgroup $\Gamma_T \subseteq \Gamma$ consisting of all elements $g \in \Gamma$ such that $g \triangleright T = T$. Then (Γ_T, T) is a matched pair by restriction (see Remark 4.2). Theorem 5.2 implies that the associated split extension $k^{\Gamma_T} \# kT$ is a quantum permutation algebra.

Remark 5.4. Observe that the conclusion in Theorem 5.2 holds in either of the following cases:
(i) F is abelian, or
(ii) the action $\triangleright \colon \Gamma \times F \to F$ is trivial, that is, H is a split cocentral extension.

We next discuss some families of examples of finite quantum permutation algebras.

Example 5.5. The dual of the Drinfeld double $D(G)$ of a finite group G fits into a central abelian exact sequence $k \to k^G \to D(G)^* \to kG \to k$. By Theorem 5.1, we get:

Corollary 5.6. *Let G be a finite group. Then $D(G)^*$ is a quantum permutation algebra and* $\mathrm{d}(D(G)^*) \leq |G|^3$.

Example 5.7. *Dimension pqr.* Let p, q and r be pairwise distinct prime numbers. A semisimple Hopf algebra H of dimension p, p^2 or pq is necessarily commutative or cocommutative, so H is a quantum permutation algebra.

It is known that every semisimple Hopf algebra H of dimension p^3 fits into a central abelian extension $k \to k^{\mathbb{Z}_p} \to H \to k(\mathbb{Z}_p \times \mathbb{Z}_p) \to k$ [18]. Hence H is a quantum permutation algebra.

Assume next that $\dim H = pqr$. By [10, Corollary 9.4], H is a split abelian extension. Such extensions are classified in [25, Section 4]; in particular, they must be either central or cocentral. It follows from Theorems 5.1 and 5.2 that H is a quantum permutation algebra.

In conclusion, we can state the following:

Proposition 5.8. *Suppose that the dimension of H divides p^3 or pqr. Then H is a quantum permutation algebra.*

Consider the case where $\dim H = pq^2$, $p \neq q$. By the results in [10, Subsection 9.2] and the classification results of abelian extensions in [25], either H or H^* fits into a central abelian exact sequence.

Proposition 5.9. *Suppose that H is nontrivial and $\dim H = pq^2$, $p \neq q$. If either $p > q$ or $p = 2$, then H is a quantum permutation algebra. If $2 < p < q$, then $q = 1 (\mathrm{mod}\ p)$ and H is a cocentral (non split) exact sequence $k \to k^{\mathbb{Z}_q \times \mathbb{Z}_q} \to H \to k\mathbb{Z}_p \to k$. In the last case, H^* is a quantum permutation algebra.*

Proof. When $p > q$, H is one of the (self-dual) central abelian extensions constructed in [11]. On the other hand, when $p = 2$, H fits into a central abelian exact sequence $k \to k^\Gamma \to H \to kF \to k$, where $\Gamma = \mathbb{Z}_q$, $F = D_q$, or $\Gamma = \mathbb{Z}_p$, $F = \mathbb{Z}_q \times \mathbb{Z}_q$. These extensions are classified in [19]. See [25, Lemmas 1.3.9 and 1.3.11]. In the case $2 < p < q$, it follows from [25, Subsection 1.4] that H fits into the prescribed exact sequence. The proposition follows from Theorem 5.1. $\square$

We point out that in the case of a non split exact sequence $k \to k^{\mathbb{Z}_q \times \mathbb{Z}_q} \to H \to k\mathbb{Z}_p \to k$, there are examples of nontrivial Hopf algebras H with no proper central Hopf subalgebra. The dual Hopf algebra H^* can also be constructed as a twisting deformation of a dual group algebra.

Example 5.10. *Dimension* 16. It follows from [13, Theorem 9.1] that every cosemisimple Hopf algebra H of dimension 16 over k fits into a central exact sequence $k \to k^{\mathbb{Z}_2} \to H \to kF \to k$, where F is group of order 8. Therefore H is a quantum permutation algebra.

Example 5.11. *Examples with irreducible characters of degree* ≤ 2. As another example (see [1]), we get that the nontrivial Hopf algebras $H = \mathcal{A}_n$ or $\mathcal{B}_n$, studied by Masuoka in [22] are quantum permutation algebras. Indeed, they fit into a central abelian exact sequence $k \to k^{\mathbb{Z}_2} \to H \to kF \to k$, where F is a dihedral group.

It follows from [8] that if H is a nontrivial semisimple Hopf algebra and $\chi \in H^*$ is a faithful self-dual irreducible character of degree 2, then H fits into a central exact sequence $k \to k^{\mathbb{Z}_2} \to H \to kF \to k$, where F is a polyhedral group. Therefore H is also a quantum permutation algebra in this case.

More generally, let H be a semisimple Hopf algebra such that its irreducible corepresentations are of dimension ≤ 2. Suppose in addition that H^* contains no proper central Hopf subalgebra. By [8, Theorem 6.4] H fits into a central abelian extension $k \to k^\Gamma \to H \to kF \to k$, with $\Gamma \neq 1$. Therefore H is a quantum permutation algebra.

6. Right coideal subalgebras in split extensions

Let $H = k^\Gamma \# kF$ be a split abelian extension. Our aim in this section is to give some restrictions on the associated actions $\triangleright$ and $\triangleleft$, in order that H contains a commutative right coideal subalgebra. In the case of right coideal subalgebras which are 'extremal' in a certain sense, we obtain conditions that correspond, roughly, to the assumptions in Theorems 5.1 and 5.2 (see Proposition 6.3). The results will be used in the next section.

Consider the canonical projection $\pi = (\epsilon \otimes \mathrm{id}) : H \to kF$. Note that $\pi(e_g \# x) = \delta_{g,1} x$, for all $g \in \Gamma$, $x \in F$.

Then H is a right kF-comodule algebra via $\rho = (\mathrm{id} \otimes \pi)\Delta : H \to H \otimes kF$. In other words, H is an F-graded algebra $H = \oplus_{x \in F} H_x$, where, for all $x \in F$,

$$H_x = \{h \in H | \, (\mathrm{id} \otimes \pi)\Delta(h) = h \otimes x\} = k^\Gamma \# x.$$

Suppose $R \subseteq H$ is a right coideal subalgebra, that is, $\Delta(R) \subseteq R \otimes H$. Then R is a kF-subcomodule algebra of H, thus it is a graded subalgebra, $R = \oplus_{x \in F} R_x$, where $R_x = R \cap H_x$, for all $x \in F$.

Let $\operatorname{Supp} R \subseteq F$ denote the *support* of R, that is, $\operatorname{Supp} R = \{x \in F |\, R_x \neq 0\}$.

Since R is a right coideal subalgebra of H, then $\pi(R) = kT$, where T is a subgroup of F. On the other hand, π defines, by restriction, an epimorphism of right kF-comodules $\pi : R \to kT$. In other words, $\pi : R \to kT$ is a (surjective) map of F-graded spaces, with respect to the natural grading on kT. Therefore

$$\pi(R_x) = \begin{cases} kx, & \text{if } x \in T, \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

Notice that, since $R_x = R \cap H_x = R \cap (k^\Gamma \# x)$, then every nonzero element of R_x, $x \in \operatorname{Supp} R$, is of the form $f \# x$, where $f \in k^\Gamma$ is nonzero.

Furthermore, if $x \in F$, then $x \in T$ if and only if there exists $f \# x \in R_x$ with $f(1) \neq 0$.

Let $\rightharpoonup : \Gamma \times k^\Gamma \to k^\Gamma$ denote the action by algebra automorphisms of Γ on k^Γ given by right translations, that is, $(s \rightharpoonup f)(g) = f(gs)$, for all $s, g \in \Gamma$.

Lemma 6.1. (i) *For all $g \in \Gamma$, $x \in F$, we have*

$$R_{g \triangleright x} = \{(g \rightharpoonup f) \# (g \triangleright x) |\; f \# x \in R_x\}.$$

(ii) $\operatorname{Supp} R$ *is a Γ-stable subset of F containing T.*

(iii) *For all $x \in \operatorname{Supp} R$, there exists $t \in \Gamma$ such that $t \triangleright x \in T$.*

Proof. (i). Let $x \in F$ and let $f \in k^\Gamma$, such that $f \# x \in R_x$. Since $\Delta(R) \subseteq R \otimes H$,

$$\begin{aligned} \Delta(f \# x) &= \sum_{g \in \Gamma} f(g) \Delta(e_g \# x) = \sum_{s,t \in \Gamma} f(st)(e_s \# t \triangleright x) \otimes (e_t \# x) \\ &= \sum_{t \in \Gamma} ((t \rightharpoonup f) \# t \triangleright x) \otimes (e_t \# x) \in R \otimes H. \end{aligned}$$

Fix $g \in \Gamma$. Evaluating the right tensorand of the last expression in $\epsilon_F \# g \in H^*$, we get that $(g \rightharpoonup f) \# (g \triangleright x) \in R$, and since this is homogeneous of degree $g \triangleright x$, then $(g \rightharpoonup f) \# (g \triangleright x) \in R_{g \triangleright x}$. This shows that $\{(g \rightharpoonup f) \# (g \triangleright x) |\; f \# x \in R_x\} \subseteq R_{g \triangleright x}$.

Since $g \in \Gamma$ was arbitrary, the other inclusion follows from this applied to $g^{-1} \in \Gamma$. This proves (i).

(ii). By (i), $\operatorname{Supp} R$ is Γ-stable. The inclusion $T \subseteq \operatorname{Supp} R$ follows from (6.1).

(iii). Let $x \in \operatorname{Supp} R$ and let $f \# x \in R_x$, where $0 \neq f \in k^\Gamma$. By (i), we have that $(t \rightharpoonup f) \# (t \triangleright x) \in R_{t \triangleright x}$, for all $t \in \Gamma$.

Since $f \neq 0$, there exists $s \in \Gamma$ such that $f(s) \neq 0$. Then $(s \rightharpoonup f)(1) = f(s) \neq 0$. Hence $s \triangleright x \in T$. This proves (iii). □

Remark 6.2. It follows from Lemma 6.1 that, for all $x \in F$, $x \in \operatorname{Supp} R$ if and only if $x^{-1} \in \operatorname{Supp} R$. Indeed, if $s \in \Gamma$ is such that $s \triangleright x \in T$, then $(s \triangleleft x) \triangleright x^{-1} = (s \triangleright x)^{-1} \in T$. Thus $(s \triangleleft x) \triangleright x^{-1} \in \operatorname{Supp} R$. Since $\operatorname{Supp} R$ is Γ-stable, it follows that $x^{-1} \in \operatorname{Supp} R$. This proves the claim.

Assume in addition that $R \subseteq H$ is a *commutative* right coideal subalgebra. Then T is an abelian subgroup of F.

Proposition 6.3. *Let $H = k^\Gamma \# kF$. Let also $R \subseteq H$ be a commutative right coideal subalgebra and let $\pi(R) = kT$, where T is an abelian subgroup of F. Then:*

(i) *If $k^\Gamma \subseteq R$, then T acts trivially on Γ via $\triangleleft$.*
(ii) *If $k^\Gamma \cap R = k1$, then T is stable under the action $\triangleright$ of Γ.*

Proof. (i). Consider the F-gradings $H = \oplus_{x\in F} H_x$, $R = \oplus_{x\in F} R_x$, as before. In this case, we have $k^\Gamma \subseteq R_1 \subseteq H_1 = k^\Gamma$. Hence $k^\Gamma = R_1 = H_1$. In particular, since R_x is an R_1-module under left multiplication, then $k^\Gamma R_x \subseteq R_x$, for all $x \in F$.

Let $f\#x \in R_x$ and let $g \in \Gamma$. Then $e_g(f\#x) = f(g)\, e_g\#x \in R_x$. Thus $R_x = k^S\#x$, for some subset $S \subseteq \Gamma$. (Note that $x \in T$ if and only if $1 \in S$.)

Let $x \in \operatorname{Supp} R$ and put $R_x = k^S\#x$, $S \subseteq \Gamma$, as above. Let $s \in S$. Since $e_s \in k^\Gamma \subseteq R$, and R is commutative, we have

$$e_s\#x = e_s(e_s\#x) = (e_s\#x)e_s = \delta_{s\triangleleft x, s} e_s\#x.$$

Therefore $s \triangleleft x = s$, for all $s \in S$.

Now suppose $x \in T$, so that $e_1\#x \in R_x$. By Lemma 6.1 (i), $R_{t\triangleright x} = k^{St^{-1}}\#(t\triangleright x)$, for all $t \in \Gamma$. In particular, $e_{t^{-1}}\#t \triangleright x = t \rightharpoonup e_1\#t \triangleright x \in R_{t\triangleright x}$, for all $t \in \Gamma$.

As we have seen above, this implies that $t^{-1} \triangleleft (t \triangleright x) = t^{-1}$, for all $t \in \Gamma$. Thus

$$1 = (t^{-1}t) \triangleleft x = \big(t^{-1} \triangleleft (t \triangleright x)\big)\, (t \triangleleft x) = t^{-1}(t \triangleleft x),$$

for all $t \in \Gamma$. This means that the action of x on Γ is trivial. Since $x \in T$ was arbitrary, this proves (i).

(ii). Suppose $x \in T$ and let $f\#x \in R_x$ such that $f(1) = 1$. Since $x^{-1} \in T$, there exists $f'\#x^{-1} \in R_{x^{-1}}$ with $f'(1) = 1$. The product $(f\#x)(f'\#x^{-1})$ belongs to $R_1 = k1$, by assumption.

On the other hand, we have $(f\#x)(f'\#x^{-1}) = f(x.f')\#1$. Thus

$$(f\#x)(f'\#x^{-1}) = f(x.f')\#1 = (f(x.f'))(1)\, 1\#1 = 1\#1,$$

since $(f(x.f'))(1) = f(1)f'(1 \triangleleft x) = f(1)f'(1) = 1$. This shows that, for all $x \in T$, R_x contains an invertible element $f\#x$ with inverse $f'\#x^{-1} \in R_{x^{-1}}$. In particular, $f \in k^\Gamma$ is invertible and therefore $f(g) \neq 0$, for all $g \in \Gamma$.

Let now $x \in \operatorname{Supp} R$. By Lemma 6.1 (iii), $s \triangleright x \in T$, for some $s \in \Gamma$. Hence, by the above, there is an invertible element $f\#s \triangleright x \in R_{s\triangleright x}$ with $f(g) \neq 0$, for all $g \in \Gamma$.

By Lemma 6.1 (i), we have $(s^{-1} \rightharpoonup f)\#x = (s^{-1} \rightharpoonup f)\#s^{-1} \triangleright (s \triangleright x) \in R_x$. Moreover, $(s^{-1} \rightharpoonup f)(1) = f(s) \neq 0$. Hence $x \in T$.

This shows that $\operatorname{Supp} R = T$ and then T is Γ-stable, by Lemma 6.1 (ii). This proves (ii) and finishes the proof of the proposition. □

Example 6.4. Let Γ be a finite group acting by automorphisms on a finite group F via $\triangleright : \Gamma \times F \to F$, and let $H = k^\Gamma \# kF$ (so $\triangleleft$ is trivial in this case). Thus H is a central split abelian extension of k^Γ.

Let $T \subset F$ be a subgroup and let

$$X(T) = \operatorname{Span}\{e_g\#g^{-1} \triangleright y,\ g \in \Gamma,\ y \in T\}.$$

Then $X(T)$ is a right coideal subalgebra of H (containing k^Γ) and it is commutative if T is abelian.

For appropriate choices of the abelian subgroup T, the algebras $X(T)$ provide examples of commutative coideal subalgebras $R \subseteq H$ with $\operatorname{Supp} R$ not necessarily included in an abelian subgroup.

7. Split extensions associated to the symmetric group

In this section we shall give examples of cosemisimple Hopf algebras H which are not quantum permutation algebras.

Let $\mathbb{S}_n$ denote the symmetric group on n symbols. Let $H = k^{C_n} \# k\mathbb{S}_{n-1}$ be the split abelian extension associated to the matched pair $(C_n, \mathbb{S}_{n-1})$ arising from the exact factorization $\mathbb{S}_n = \Gamma F$, where $\Gamma = C_n = \langle z \rangle \simeq \mathbb{Z}_n$, $z = (12 \dots n)$, and $F = \{x \in \mathbb{S}_n | \, x(n) = n\} \simeq \mathbb{S}_{n-1}$. (Actually, H is the unique, up to isomorphism, Hopf algebra fitting into an exact sequence $k \to k^{\mathbb{Z}_n} \to H \to k\mathbb{S}_{n-1} \to k$ [20, Theorem 4.1].)

Remark 7.1. It follows from Theorem 5.2 that $H^* = k^{\mathbb{S}_{n-1}} \# kC_n$ is a quantum permutation algebra. Note that, as for Drinfeld doubles, we have $D(H) \simeq D(H^*)$, and $D(H)^* \simeq (D(\mathbb{S}_n)^*)^\sigma$ is a twisting deformation of $D(\mathbb{S}_n)^*$, for a certain convolution invertible cocycle $\sigma : D(\mathbb{S}_n)^* \otimes D(\mathbb{S}_n)^* \to k$ [5].

We quote the following fact on stabilizers, that follows from [12, Lemma 3.2].

Lemma 7.2. *We have $F_1 = \mathbb{S}_{n-1}$ and $F_{z^j} = \{x \in \mathbb{S}_{n-1} | \, x(n-j) = n-j\} \simeq \mathbb{S}_{n-2}$, $1 \leq j \leq n-1$. In particular, the only subgroup of $F = \mathbb{S}_{n-1}$ that acts trivially on C_n is the trivial subgroup $\{1\}$.*

Proof. It is shown in [12, Lemma 3.2] that there are two orbits for the action of $\mathbb{S}_{n-1}$ on C_n, namely $\mathcal{O}_1 = \{1\}$ and $\mathcal{O}_z = \{z, \dots, z^{n-1}\}$. We have $F_1 = \mathbb{S}_{n-1}$ and $F_z = \{x \in \mathbb{S}_{n-1} | \, x(n-1) = n-1\} \simeq \mathbb{S}_{n-2}$.

Moreover, for each $1 \leq j \leq n-1$, $z^j = z \triangleleft x_j$, where x_j is the transposition $x_j = (n-1 \, n-j)$. Therefore, $F_{z^j} = x_j^{-1} F_z x_j$ is the claimed subgroup of $F = \mathbb{S}_{n-1}$. □

Suppose that $n = p$ is a prime number. Let $R \subseteq H$ be a commutative right coideal subalgebra, and let $\pi(R) = kT$, where $T \subseteq \mathbb{S}_{p-1}$ is an abelian subgroup. Since $R_1 = R \cap k^\Gamma$ is a right coideal subalgebra (hence a Hopf subalgebra) of k^Γ, then $\dim R_1$ divides $|\Gamma| = p$. Therefore, we must have either $R \cap k^\Gamma = k^\Gamma$ or $R \cap k^\Gamma = k1$.

By Proposition 6.3, in the second case T is C_p-stable, while in the first case T acts trivially on $\Gamma = C_p$, and thus $T = 1$, in view of Lemma 7.2. This implies that in this case $R \subseteq k^{C_p}$, by Lemma 6.1 (iii).

Lemma 7.3. *Let p be a prime number and suppose that $H = k^{C_p} \# k\mathbb{S}_{p-1}$ is a quantum permutation algebra. Then $\mathbb{S}_{p-1}$ is generated by its abelian C_p-stable subgroups.*

Proof. The assumption implies that H is generated by commutative right coideal subalgebras $R^1, \dots, R^N$, $N \geq 1$. Letting $kT^j = \pi(R^j)$, we have that the abelian subgroups T^j, $1 \leq j \leq N$, generate $F = \mathbb{S}_{p-1}$.

By the above, either $T^j = 1$ or T^j is C_p-stable. Hence $\mathbb{S}_{p-1}$ is actually generated by abelian C_p-stable subgroups, as claimed. □

We can now state the main result of this section:

Theorem 7.4. *The cosemisimple Hopf algebras $H = k^{C_4}\#k\mathbb{S}_3$ and $H = k^{C_5}\#k\mathbb{S}_4$ are* not *quantum permutation algebras.*

In particular, there exist finite dimensional cosemisimple Hopf algebras which are not quantum permutation algebras.

Proof. Let $H = k^{C_5}\#k\mathbb{S}_4$. In this case, the action of $C_5 = \langle(12345)\rangle$ on $\mathbb{S}_4$ is written down explicitly in Table 1 of [12, pp. 15]. It turns out that the only abelian C_5-stable subgroups of $\mathbb{S}_4$ are contained in the cyclic subgroup $\langle(1342)\rangle$. Thus they do not generate $\mathbb{S}_4$. Lemma 7.3 implies that H is not a quantum permutation algebra.

Let now $H = k^{C_4}\#k\mathbb{S}_3$. In this case, the action of C_4 on $\mathbb{S}_3$ has three orbits:

$$\{1\}, \quad \{(13)\}, \quad \{(12), (23), (123), (132)\}. \tag{7.1}$$

In particular, the only abelian subgroup of $\mathbb{S}_3$ which is C_4-stable is $\langle(13)\rangle \simeq \mathbb{Z}_2$.

We have in addition that $kG(H) = k^{C_4}\#k\langle(13)\rangle$, and $G(H) \simeq D_4$.

Let $R \subseteq H$ be a commutative right coideal subalgebra. As before, consider the $\mathbb{S}_3$-grading $R = \oplus_{x\in\mathbb{S}_3} R_x$, where $R_x = R \cap (k^{C_4}\#x)$ and let $\pi(R) = kT$, where $T \subseteq \mathbb{S}_3$ is an abelian subgroup.

The subalgebra $R_1 = R \cap k^{C_4}$ is a right coideal subalgebra (hence a Hopf subalgebra) of k^{C_4}. Then either $R_1 = k1$ or $R_1 = k^{C_4}$, or $R_1 = k^{C_4/L}$, where $L = \{1, z^2\}$ is the only subgroup of order 2 of C_4.

By Proposition 6.3, the assumption $R_1 = k^{C_4}$ implies that T acts trivially on C_4, and thus $T = 1$, by Lemma 7.2. Also, if $R_1 = k1$, then T is C_4-stable and therefore $T \subseteq \langle(13)\rangle$.

In any of these cases, we obtain that $\operatorname{Supp} R \subseteq \langle(13)\rangle$, by Lemma 6.1 (iii), and thus $R \subseteq kG(H)$.

Suppose that there exists a commutative right coideal subalgebra R such that $\operatorname{Supp} R \nsubseteq \langle(13)\rangle$. By Proposition 6.3 (iii), also $T \subsetneq \langle(13)\rangle$ and $\operatorname{Supp} R \neq \mathbb{S}_3$ (otherwise, the transposition (13) would belong to the orbit of another cycle in $\mathbb{S}_3$).

By the above, $R_1 = k^{C_4/L}$ is of dimension 2 and in view of (7.1),

$$\operatorname{Supp} R = \{1\} \cup \{(12), (23), (123), (132)\}, \tag{7.2}$$

since by Proposition 6.3 (ii), $\operatorname{Supp} R$ is C_4-stable.

The subalgebra $R_1 = k^{C_4/L}$ of k^{C_4} is spanned by the idempotents $e_L = e_1 + e_{z^2}$ and $e_{Lz} = e_z + e_{z^3}$.

For a subset $X \subseteq C_4$ and $f \in k^{C_4}$, let us denote $f_X = \sum_{x\in X} f(x)e_x$. Note that $f_L = e_L f$, for all $f \in k^{C_4}$.

Let $x \in \operatorname{Supp} R$ and let $f \in k^{C_4}$ such that $f\#x \in R_x$. Since R is commutative by assumption, we have

$$e_L(f\#x) = f_L\#x = (f\#x)e_L = f_{L\triangleleft x^{-1}}\#x, \tag{7.3}$$

that is, $f_L = f_{L\triangleleft x^{-1}}$. Hence, if $x \neq 1$, then $f(z^2) = 0$. Otherwise, $z^2 \triangleleft x = z^2$, implying that $x \in F_{z^2} = \langle(13)\rangle$ (see Lemma 7.2), which contradicts (7.2).

Now suppose that $x \in T$, $x \neq 1$. Then there exists $f\#x \in R_x$ such that $f(1) = 1$. Thus $f_L = e_1$, by the above. In particular, $e_1\#x = e_L(f\#x) \in R_x$, and by Proposition 6.3 (i), $e_t\#t^{-1} \triangleright x \in R_{t^{-1}\triangleright x}$, for all $t \in C_4$.

By (7.3), this implies that $(e_t)_L = (e_t)_{L \triangleleft (t^{-1} \triangleright x)^{-1}}$, for all $t \in C_4$. In particular, taking $t = z^2$, we get that $z^2 \in L \triangleleft (z^2 \triangleright x)^{-1}$, so that $z^2 \triangleright x \in F_{z^2} = \langle (13) \rangle$. This contradicts again (7.2), since $1 \neq z^2 \triangleright x \in \operatorname{Supp} R$.

This shows that there can exist no commutative right coideal subalgebra R with $\operatorname{Supp} R \nsubseteq \langle (13) \rangle$. Then we conclude that every commutative right coideal subalgebra of H is contained in $kG(H)$. Therefore H is not a quantum permutation algebra, by Theorem 2.3. This finishes the proof of the theorem. □

Remark 7.5. The results in Section 5 ensure that any semisimple Hopf algebra of dimension less than or equal to 23 is a quantum permutation algebra. Theorem 7.4 implies that this bound is optimal, since $H = k^{C_4} \# k\mathbb{S}_3$ is a non quantum permutation algebra of dimension 24. In particular, 24 *is the smallest possible dimension that a non quantum permutation algebra can have.*

Remark 7.6. Observe that in the case where $H = k^{C_5} \# k\mathbb{S}_4$, the exact factorization $\mathbb{S}_5 = \mathbb{S}_4 C_5$ restricts to an exact factorization $\mathbb{A}_5 = \mathbb{A}_4 C_5$. We may therefore consider the split extension $H' = k^{C_5} \# k\mathbb{A}_4$. (Indeed, H' is isomorphic to a Hopf subalgebra of H, by Remark 4.2.)

The arguments used so far apply *mutatis mutandi* to this new matched pair. Then, as before, we get that H' is not a quantum permutation algebra. The example provided by H' has dimension 60.

Remark 7.7. Note that if H is not a quantum permutation algebra, then the tensor product $\tilde{H} = H \otimes H^*$ is not a quantum permutation algebra neither (since H is a quotient of $\tilde{H}$), and $\tilde{H}$ is *self-dual*. Theorem 7.4 implies that there exist self-dual cosemisimple Hopf algebras which are not quantum permutation algebras.

Remark 7.8. As noted before, for H as in Theorem 7.4, we have $D(H)^* \simeq (D(\mathbb{S}_n)^*)^\sigma$ ($n = 4$ or 5) is a twisting deformation of a quantum permutation algebra (see Corollary 5.6 and Remark 7.1). Since H is a quotient of $D(H)^*$, then $D(H)^*$ is not a quantum permutation algebra.

This provides an example of a twisting deformation of a quantum permutation algebra which is not a quantum permutation algebra.

As a consequence of Theorem 7.4, and since H^* is a quantum permutation algebra, we get the following:

Corollary 7.9. *The class of quantum permutation algebras in not stable neither under duality nor under Hopf algebra extensions nor under twisting deformations.*

8. Quantum permutation algebras obtained from twisting

We have seen in the previous section that the class of quantum permutation algebras is not stable under twisting. We begin by giving a stability result (Proposition 8.1) for twistings of quantum permutation algebras, under a technical condition on the cocycle. Then we give (Theorem 8.3) a construction of quantum permutation algebras by the twisting of certain linear algebraic groups, using the results from [3].

The results combined together cover the known quantum permutation algebras obtained by twisting.

Proposition 8.1. *Let H be a quantum permutation algebra generated by the coefficients of a magic matrix $x = (x_{ij}) \in M_n(H)$. Let $\sigma : H \otimes H \longrightarrow k$ be a 2-cocycle satisfying*

$$\sigma(x_{ij}, x_{il}) = \delta_{ij}\delta_{il}, \forall i, j, l$$

Then H^σ is a quantum permutation algebra.

Proof. Recall that the Hopf algebra H^σ is H as a coalgebra, and the product is defined by

$$[x][y] = \sigma(x_1, y_1)\sigma^{-1}(x_3, y_3)[x_2y_2], \quad x, y \in H,$$

where an element $x \in H$ is denoted $[x]$, when viewed as an element of H^σ. Then

$$\begin{aligned}
[x_{ij}][x_{il}] &= \sum_{r,s,p,q} \sigma(x_{ir}, x_{is})\sigma^{-1}(x_{pj}, x_{ql})[x_{rp}x_{sq}] \\
&= \sum_{p,q} \sigma^{-1}(x_{pj}, x_{ql})[x_{ip}x_{iq}] \\
&= \sum_{p} \sigma^{-1}(x_{pj}, x_{pl})[x_{ip}] \\
&= \delta_{jl}[x_{ij}].
\end{aligned}$$

since we also have $\sigma^{-1}(x_{ij}, x_{il}) = \delta_{ij}\delta_{il}, \forall i, j, l$. As $\sum_i [x_{ij}] = [1] = \sum_i [x_{ji}]$, we conclude that $([x_{ij}])$ is a magic matrix and hence H^σ is a quantum permutation algebra. □

Example 8.2. The twistings of $k^{\mathbb{S}_n}$ in [6] are of this type.

Let Γ be an abelian group and let $\sigma \in Z^2(\Gamma, k^*)$. Recall that the character group $\widehat{\Gamma}$ acts faithfully by automorphisms on the twisted group algebra $k_\sigma\Gamma$ by

$$\chi.g = \chi(g)g, \ \forall \chi \in \widehat{\Gamma}, g \in \Gamma$$

So we consider $\widehat{\Gamma}$ as a subgroup of $\mathrm{Aut}(k_\sigma\Gamma)$.

Theorem 8.3. *Let Γ be an abelian group and let $\sigma \in Z^2(\Gamma, k^*)$. Consider a linear algebraic group G such that $\widehat{\Gamma} \subseteq G \subseteq \mathrm{Aut}(k_\sigma\Gamma)$. Then σ induces a 2-cocycle σ' on $\mathcal{O}(G)$ such that $\mathcal{O}(G)^{\sigma'}$ is a quantum permutation algebra.*

Proof. The cocycle σ' is constructed in the standard way: the inclusion $\widehat{\Gamma} \subset G$ induces a surjective Hopf algebra map $\mathcal{O}(G) \to k^{\widehat{\Gamma}}$ which, composed with the canonical isomorphism $k^{\widehat{\Gamma}} \simeq k\Gamma$ yields a surjective Hopf algebra map $p : \mathcal{O}(G) \longrightarrow k\Gamma$. The 2-cocycle σ' is defined by $\sigma' = \sigma(p \otimes p)$.

Now let $A_{aut}(k_\sigma\Gamma)$ be the universal Hopf algebra coacting on $k_\sigma\Gamma$ and leaving the canonical trace invariant (see [29, 3]). Since the automorphism group $\mathrm{Aut}(k_\sigma\Gamma)$ preserves the canonical trace, the universal property of $A_{aut}(k_\sigma\Gamma)$ yields a Hopf algebra map $q : A_{aut}(k_\sigma\Gamma) \to \mathcal{O}(G)$. Now the composition of surjective Hopf algebra maps

$$A_{aut}(k_\sigma\Gamma) \xrightarrow{q} \mathcal{O}(G) \xrightarrow{p} k\Gamma$$

yields a composition of surjective Hopf algebra maps

$$A_{aut}(k_\sigma\Gamma)^{\sigma''} \xrightarrow{q} \mathcal{O}(G)^{\sigma'} \xrightarrow{p} k\Gamma,$$

where $\sigma'' = \sigma(pq \otimes pq)$. We know from [3] that $A_{aut}(k_\sigma\Gamma) \simeq A_{aut}(k\Gamma)^{\sigma^{-1}(\pi\otimes\pi)}$ where $\pi : A_{aut}(k\Gamma) \longrightarrow k\Gamma$ is the Hopf algebra map arising from the coaction of $k\Gamma$

on itself. It is then clear that $A_{aut}(k_\sigma\Gamma)^{\sigma''} \simeq A_{aut}(k\Gamma)$. The latter is a quantum permutation algebra since $k\Gamma$ is commutative, and hence we conclude that so is $\mathcal{O}(G)^{\sigma'}$. □

Remark 8.4. The cocycle σ' in Theorem 8.3 is the one 'lifted' from the cocycle σ on $\widehat{\Gamma} \simeq \Gamma$.

Remark 8.5. If $k_\sigma\Gamma$ is non commutative and if the only subgroup of $\widehat{\Gamma}$ that is normal in G is trivial, then the algebra $\mathcal{O}(G)^{\sigma'}$ is non commutative (see [26]).

Example 8.6. Let $\Gamma = \mathbb{Z}_2^n$, $n \geq 2$, and consider the bicharacter σ on Γ given by $\sigma(t_i, t_j) = -1$, if $i < j$, $\sigma(t_i, t_j) = 1$, if $i \geq j$, where t_i, $1 \leq i \leq n$, denote the standard generators of Γ. In this case the twisted group algebra $k_\sigma\Gamma$ is isomorphic to the Clifford algebra $Cl_n(k) = k[t_i, 1 \leq i \leq n|\ t_i^2 = 1, t_it_j = -t_jt_i, i \neq j]$. Since the orthogonal group $O_n(k)$ acts naturally on $Cl_n(k)$ by algebra automorphisms, we get from Theorem 8.3 that for any subgroup $\mathbb{Z}_2^n \subseteq G \subseteq O_n(k)$, the cocycle twist $\mathcal{O}(G)^{\sigma'}$ is a quantum permutation algebra.

When $G = O_n(k)$, we get the hyperoctahedral quantum group $O_n^{-1}(k)$ from [2].

Example 8.7. Let $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_n$ and let σ be the 2-cocycle on Γ given by $\sigma((i,j),(t,l)) = w^{jt}$, where $w \in k^*$ is a primitive n-th root of unity. In this case we have $k_\sigma\Gamma \simeq M_n(k)$, so that $\mathrm{Aut}(k_\sigma\Gamma) \simeq \mathrm{PGL}_n(k)$.

By Theorem 8.3, for every linear algebraic group G such that $\widehat{\Gamma} \subseteq G \subseteq \mathrm{PGL}_n(k)$, the cocycle twist $\mathcal{O}(G)^{\sigma'}$ is a quantum permutation algebra. The twisted examples from [1] are of this type for $n = 2$.

9. Quantum permutation envelope

Let H be a cosemisimple Hopf algebra. Consider the subalgebra $H_{qp} \subseteq H$ generated by the matrix coefficients of all separable commutative (right *and* left) coideal subalgebras of H.

Lemma 9.1. *H_{qp} is a Hopf subalgebra of H containing all quantum permutation algebras $A \subseteq H$.*

Proof. It is clear that H_{qp} is a subbialgebra. Since the image of a separable commutative right (respectively, left) coideal subalgebra under the antipode of H is a separable commutative left (respectively, right) coideal subalgebra, then we have $\mathcal{S}(H_{qp}) = H_{qp}$. Then H_{qp} is a Hopf subalgebra.

Since every quantum permutation algebra $A \subseteq H$ is generated by matrix coefficients of some separable commutative left coideal subalgebras, then $A \subseteq H_{qp}$. This finishes the proof of the lemma. □

Definition 9.2. The Hopf subalgebra H_{qp} will be called the *quantum permutation envelope* of H.

Proposition 9.3. *Suppose that H is finite dimensional and cosemisimple. Then H_{qp} is the maximal quantum permutation algebra contained in H. Moreover, H_{qp} is generated as an algebra by the matrix coefficients of all separable commutative right (or left) coideal subalgebras of H.*

Proof. Note that, being finite dimensional, the subbialgebra H' generated as an algebra by the matrix coefficients of all separable commutative right (or left) coideal subalgebras of H is a Hopf subalgebra. This implies that $H_{qp} = H'$.

It follows from Theorem 2.3 that H_{qp} is a quantum permutation algebra, and it is maximal by Lemma 9.1. This proves the proposition. □

Example 9.4. Suppose $H = \mathcal{R}(G)$ is the algebra of representative function on a compact group G. Then the quantum permutation envelope $H_{qp} \subseteq H$ coincides with $\mathcal{R}(G/N)$ where N is the intersection of all closed normal subgroups of G of finite index.

If G is connected, the only such subgroup is G, so that $H_{qp} = k1$.

On the other extreme, the condition $G/N = G$ means exactly that G is *residually finite*, that is, morphisms from G to finite groups separate points of G.

In particular, if G is a *profinite group* (equivalently, a totally discontinuous compact group [27, I.1]), then G is residually finite.

Regarding split abelian extensions, we have:

Proposition 9.5. *Let p be a prime number and let $H = k^{\mathbb{Z}_p}\#kF$, where F is a finite group. Then $H_{qp} \subseteq H$ is the Hopf subalgebra generated, as an algebra, by $k^{\mathbb{Z}_p}\#kF'$ and $k^{\mathbb{Z}_p}\#kF''$, where F' is the largest subgroup of F acting trivially on $\mathbb{Z}_p$, and F'' is the subgroup generated by the abelian $\mathbb{Z}_p$-stable subgroups of F.*

The subgroup F' is the kernel of the homomorphism $F \to \mathbb{S}_{p-1}$ induced by $\triangleleft$.

Proof. By Remark 4.2, both F' and F'' are $\mathbb{Z}_p$-stable subgroups of F, and the bicrossed products $k^{\mathbb{Z}_p}\#kF'$ and $k^{\mathbb{Z}_p}\#kF''$ are Hopf subalgebras of H. Since F' acts trivially on $\mathbb{Z}_p$, then $k^{\mathbb{Z}_p}\#kF'$ is a central extension.

By Theorem 5.1 and Theorem 5.2, $k^{\mathbb{Z}_p}\#kF'$ and $k^{\mathbb{Z}_p}\#kF''$ are both quantum permutation algebras. Hence the subalgebra $\tilde{H} \subseteq H$ generated by them is a quantum permutation algebra, and therefore $\tilde{H} \subseteq H_{qp}$. On the other hand, by Proposition 6.3, $H_{qp} \subseteq \tilde{H}$. This proves $H_{qp} = \tilde{H}$, as claimed. □

As an example, let $H = k^{C_5}\#k\mathbb{S}_4$ be the Hopf algebra in Theorem 7.4. Then the quantum permutation envelope of H is the split extension $H_{qp} = k^{C_5}\#k\langle(1342)\rangle$, which is a cocommutative Hopf subalgebra (indeed, the action of C_5 on $\langle(1342)\rangle$ is trivial, as follows from [12]), with $\dim H_{qp} = 20$.

It follows from the definition of the actions that $H_{qp} \simeq kG$, where $G \simeq \mathbb{F}_5 \rtimes \mathbb{F}_5^*$.

We have seen in the proof of Theorem 7.4 that if $H = k^{C_4}\#k\mathbb{S}_3$, then every commutative right coideal subalgebra of H is contained in $kG(H)$. Hence, in this case, $H_{qp} = kG(H) \simeq kD_4$.

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T. Banica: Département de Mathématiques, Université Cergy-Pontoise, 95000 Cergy-Pontoise, France.
E-mail address: `teodor.banica@u-cergy.fr`

J. Bichon: Laboratoire de Mathématiques, Université Blaise Pascal, 63177 Aubière Cedex, France.
E-mail address: `Julien.Bichon@math.univ-bpclermont.fr`

S. Natale: Fa.M.A.F. Universidad Nacional de Córdoba. CIEM – CONICET. (5000) Córdoba, Argentina.
E-mail address: `natale@famaf.unc.edu.ar`